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# Dynamic Thermoelastic Analysis of a Slab Using Finite Integral Transformation Method

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**In this work the dynamic thermoelastic response of a slab with finite thickness subjected to aerodynamic heating is investigated. The system of generalized governing equations with generalized initial and boundary conditions is solved by the method of finite integral transformation, and the analytic expressions of the transient temperature and dynamical stresses in the slab are obtained. The finite integral transformation method shows its advantage and convenience to deal with the complex boundary conditions of a dynamic thermoelastic problem. Moreover, by introducing the aerodynamic heating and taking into account a typical Mach number curve for hypersonic flight, the analytical solution is obtained to simulate the dynamic thermoelastic response of the slab in hypersonic flight environment. Calculations are carried out for zirconium diboride to obtain transient temperature and dynamical stresses induced by aerodynamic heating.**

## Nomenclature

| | | |
|---|---|---|
| $A_r$, $B_r$, $\Delta_r$ | = | coefficients in the $R(z, t)$ expression |
| $A_v$, $B_v$ | = | coefficients in the $V(z, t)$ expression |
| $c_p$ | = | specific heat at constant pressure |
| $c_v$ | = | specific heat at constant volume |
| $E$ | = | Young's modulus |
| $F(z)$ | = | initial temperature distribution |
| $f_i(t)$ | = | thermal load, $i = 1, 2$ |
| $g(z, t)$ | = | volumetric heat source function |
| $k$ | = | thermal conductivity |
| $l$ | = | slab thickness |
| $M_\infty$ | = | Mach number of the freestream |
| $N(\beta_n)$, $N(\gamma_m)$ | = | norm |
| $q_h$ | = | heat transfer due to aerodynamic heating |
| $q_c$ | = | heat transfer due to active cooling |
| $R$, $\theta$ | = | part of temperature distribution |
| $r$ | = | recovery factor |
| $T$ | = | temperature distribution |
| $T_r$ | = | recovery temperature |
| $T_\infty$ | = | temperature of the freestream |
| $T_c$ | = | temperature of the coolant |
| $t$ | = | time |
| $\bar{W}$ | = | integral transformation of $W$ |
| $W$, $V$ | = | part of displacement in the $z$ direction |
| $w$ | = | displacement in the $z$ direction |
| $z$, $x$, $y$ | = | Cartesian coordinates |
| $\alpha$ | = | coefficient of linear thermal expansion |
| $\beta_n$, $\gamma_m$ | = | eigenvalue |
| $\Gamma$ | = | changed temperature above initial condition |
| $\gamma$ | = | ratio of specific heats |
| $\kappa$ | = | thermal diffusivity |
| $\lambda$, $\mu$ | = | Lame's constant |
| $\nu$ | = | Poisson's ratio |
| $\rho$ | = | density |
| $\sigma$, $\varepsilon$ | = | stress and strain |
| $\upsilon$ | = | velocity of irrotational wave |
| $\zeta_i$ | = | heat transfer coefficient, $i = 1, 2$ |
| $\bar{\theta}$ | = | integral transformation of $\theta$ |
| $\varphi_n(z)$, $\phi_m(z)$ | = | eigenfunction |

## I. Introduction

AIRCRAFT and spacecraft structures designed for supersonic and hypersonic flight are subjected to severe aerodynamic heating during launch and reentry phases of their operations, which is caused by the air in the boundary layer being progressively slowed down. This process generates heat and consequently all external surfaces on the aircraft are heated. This in turn leads to nonuniform transient temperatures that produce dynamic thermal stresses and deformations.

Numerous investigations have been devoted to the dynamic problem of thermoelasticity since the appearance of Danilovskaya's [1,2] original papers, where he solved the problem of dynamic thermal stresses in an elastic half-space. Sneddon and Chadwick [3], Norwood and Warren [4], Lord and Shulman [5], El-Naggar and Abd-Alla [6] have considered the inertia effects in thermoelasticity. Most works deal with the problems of infinite or semi-infinite regions, however, few investigations are done with the dynamic thermal stresses in a plate with finite width. Among these, Takeuti and Furukawa [7] examined a rigorous treatment to find the exact solutions for thermal shock problems in a plate considering the presence of an inertia term or a thermoelastic coupling term, using the Laplace transform method. The thermal shock resistance of a brittle solid was analyzed by Lu and Fleck [8] for an orthotropic plate suddenly exposed to a convective medium of different temperature, where the transient temperature distribution was solved by a standard separation of variables technique, but the stresses were obtained by quasi-static process. Recently, El-Naggar et al. [9] considered the one-dimensional dynamic thermal stresses in an infinite elastic slab, solving by means of a finite difference method. However, almost all researches only deal with the symmetry boundary conditions on the two surfaces of plate.

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Nowadays there are many analytical and numerical methods of the solutions of dynamic problems, which are in the form of partial differential equations (PDEs), including the variable separation method, Laplace transform method, the orthogonal expansion technique, the Green's function approach, the finite difference method and finite element method [10]. The Laplace transform method is used widely, but it can only obtain the analytical solution for the partial differential equations with simple boundary conditions. Otherwise, the numerical method will be conducted for inverse Laplace transform. However, the finite integral transform method is more appropriate for PDEs with complex boundary conditions. This method is systematically used by Ozisik [11] to study the transient heat conduction problems, and to the best of our knowledge, it has not been applied to dynamic elastic problems.

In this work the dynamic thermoelastic response of an elastic slab with finite thickness is investigated. The system of generalized governing equations with generalized initial and boundary conditions is solved by the method of finite integral transformation and the analytic expressions of the temperature and stresses in the slab are obtained. Compared with the Laplace transform method for the analysis of dynamic thermoelastic problem, the analysis here is completely explicit solution without any implicit term. The present method can be easily employed for other combinations of boundary conditions. Moreover, by using the classical expression of the aerodynamic heating and taking account of a typical Mach number curve for hypersonic flight, the exactly analytical solution is obtained to simulate the dynamically thermoelastic response of the slab in hypersonic flight environment. Calculations are carried out for zirconium diboride ($ZrB_2$) ceramic, which is the representative of ultrahigh temperature ceramics (UHTC), to investigate the transient temperature and dynamical stresses induced by aerodynamic heating.

## II. Dynamic Thermoelastic Analysis of an Infinite Slab

An infinitely long elastic slab with thickness $2l$ is considered, with Cartesian coordinates embedded at the center of the slab, as shown in Fig. 1. The slab is static and has a temperature distribution $F(z)$ at the initial state. At time $t = 0$, the top ($z = -l$) and bottom ($z = l$) surfaces are instantaneously subjected to thermal shock. This in turn leads to nonuniform transient temperatures that produce thermal stresses and deformations. In the following, we firstly analyze the transient temperature, and then present the dynamical stress distribution by use of finite integral transform method. All thermal and physical properties are assumed to be constant, that is, independent of temperature.

### A. Transient Heat Transfer Problem

Assuming that the temperature distribution obeys the one-dimensional heat conduction equation as follows:

$$\frac{\partial^2 T}{\partial z^2} + \frac{1}{k} g(z,t) = \frac{1}{\kappa}\frac{\partial T}{\partial t} \qquad -l < z < l, t > 0 \tag{1}$$

where $T(z,t)$ represents the temperature distribution at time $t$. $g(z,t)$ is the volumetric heat source function. $k$ and $\kappa$ denote thermal conductivity and thermal diffusivity, respectively. And $\kappa = k/\rho c_p$ where $\rho$ is the density and $c_p$ is the specific heat at constant pressure of the solid. Equation (1) is subjected to the following initial and boundary conditions:

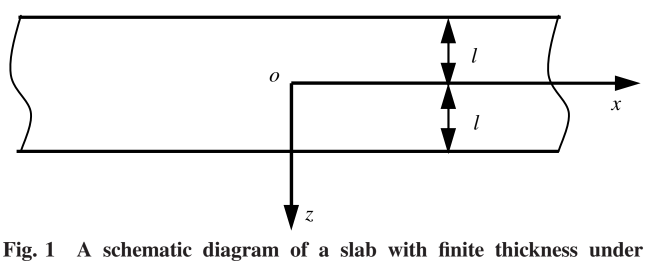


**Fig. 1 A schematic diagram of a slab with finite thickness under thermal shock.**

$$T(z,t) = F(z), \quad \text{at } t = 0 \tag{2}$$

$$-k\frac{\partial T}{\partial z} + \zeta_1 T = f_1(t), \quad \text{at } z = -l \tag{3a}$$

$$k\frac{\partial T}{\partial z} + \zeta_2 T = f_2(t), \quad \text{at } z = l \tag{3b}$$

where $\zeta_i (i = 1, 2)$ are the heat transfer coefficients and $f_i(t)$ ($i = 1$, 2) are specified functions at the top ($i = 1$) and bottom ($i = 2$) surfaces, respectively.

Equations (1–3) can be solved by the method of finite integral transformation [11]. However, to avoid the nonuniformly converging solution induced by the inhomogeneous boundary condition (3), it is convenient to divide the solution of $T(z,t)$ into two parts at first as following

$$T(z,t) = \theta(z,t) + R(z,t) \qquad -l < z < l, t > 0 \tag{4}$$

where $R(z,t)$ satisfies the following inhomogeneous boundary conditions

$$-k\frac{\partial R}{\partial z} + \zeta_1 R = f_1(t), \quad \text{at } z = -l \tag{5a}$$

$$k\frac{\partial R}{\partial z} + \zeta_2 R = f_2(t), \quad \text{at } z = l \tag{5b}$$

and thus $\theta(z,t)$ satisfies the following inhomogeneous heat conduction equation

$$\frac{\partial^2 \theta}{\partial z^2} + g^*(z,t) = \frac{1}{\kappa}\frac{\partial \theta}{\partial t} \qquad -l < z < l, t > 0 \tag{6}$$

where

$$g^*(z,t) = \frac{\partial^2 R}{\partial z^2} - \frac{1}{\kappa}\frac{\partial R}{\partial t} + \frac{1}{k} g(z,t) \tag{7}$$

and the following initial and homogeneous boundary conditions

$$\theta(z,t) = F(z) - R(z,t) \equiv F^*(z), \quad \text{at } t = 0 \tag{8}$$

$$-k\frac{\partial \theta}{\partial z} + \zeta_1 \theta = 0, \quad \text{at } z = -l \tag{9a}$$

$$k\frac{\partial \theta}{\partial z} + \zeta_2 \theta = 0, \quad \text{at } z = l \tag{9b}$$

According to the boundary condition (5), $R(z,t)$ can be selected in the form

$$R(z,t) = A_r z + B_r \tag{10}$$

where

$$A_r = \frac{\zeta_1 f_2 - \zeta_2 f_1}{\Delta_r}, \qquad B_r = \frac{k(f_1 + f_2) + l(\zeta_1 f_2 + \zeta_2 f_1)}{\Delta_r} \tag{11}$$

with

$$\Delta_r = k(\zeta_1 + \zeta_2) + 2\zeta_1\zeta_2 l \tag{12}$$

$\theta(z,t)$ is solved by the method of finite integral transformation. Guided by a procedure outlined by Ozisik [11], we now define the integral transform pair in the space variable $z$ for the solution $\theta(z,t)$ as following.

Integral-transform

$$\bar{\theta}(\beta_n, t) = \int_{-l}^{l} \varphi_n(z)\theta(z,t)\,dz \tag{13a}$$

Formula

$$\theta(z,t)=\sum_{n=1}^{\infty}\frac{\varphi_n(z)}{N(\beta_n)}\bar{\theta}(\beta_n,t) \tag{13b}$$

where eigenfunctions $\varphi_n(z)$ satisfy the following eigenvalue problem:

$$\frac{d^2\varphi}{dz^2}+\beta^2\varphi(z)=0 \qquad -l<z<l \tag{14}$$

with the following boundary conditions:

$$-k\frac{d\varphi}{dz}+\zeta_1\varphi=0, \quad \text{at } z=-l \tag{15a}$$

$$k\frac{d\varphi}{dz}+\zeta_2\varphi=0, \quad \text{at } z=l \tag{15b}$$

The solutions of the above eigenvalue problem are

$$\varphi_n(z)=\zeta_1\sin\beta_n(z+l)+k\beta_n\cos\beta_n(z+l) \tag{16}$$

where eigenvalues $\beta_n$ are the positive roots of the transcendental equation

$$\tan 2\beta_n l=\frac{(\zeta_1+\zeta_2)k\beta_n}{k^2\beta_n^2-\zeta_1\zeta_2} \tag{17}$$

The eigenfunction (16) satisfies the following orthogonality relation:

$$\int_{-l}^{l}\varphi_n(z)\varphi_m(z)\,dz=\begin{cases}0 & n\neq m\\ N(\beta_n) & n=m\end{cases} \tag{18}$$

where the norm $N(\beta_n)$ is defined as

$$N(\beta_n)=\int_{-l}^{l}[\varphi_n(z)]^2\,dz \tag{19}$$

We now apply the transformation (13a) to the differential Eq. (6). From the initial and boundary conditions (8) and (9) and by the mathematical derivation, this transformation finally results in

$$\frac{d\bar{\theta}}{dt}+\kappa\beta_n^2\bar{\theta}(\beta_n,t)=A(\beta_n,t) \qquad t>0 \tag{20}$$

where

$$A(\beta_n,t)=\kappa\int_{-l}^{l}\varphi_n(z)g^*(z,t)\,dz \tag{21}$$

with the following initial condition

$$\bar{\theta}(\beta_n,t)=\bar{F}^*(\beta_n), \quad \text{at } t=0 \tag{22}$$

where

$$\bar{F}^*(\beta_n)=\int_{-l}^{l}\varphi_n(z)F^*(z)\,dz \tag{23}$$

The solution of Eq. (20) subjected to the initial condition (22) gives

$$\bar{\theta}(\beta_n,t)=e^{-\kappa\beta_n^2 t}\left[\bar{F}^*(\beta_n)+\int_0^t e^{\kappa\beta_n^2\tau}A(\beta_n,\tau)\,d\tau\right] \tag{24}$$

By introducing this integral transform into the inversion formula (13b), we obtain

$$\theta(z,t)=\sum_{n=1}^{\infty}\frac{\varphi_n(z)}{N(\beta_n)}e^{-\kappa\beta_n^2 t}\left[\bar{F}^*(\beta_n)+\int_0^t e^{\kappa\beta_n^2\tau}A(\beta_n,\tau)\,d\tau\right] \tag{25}$$

Then, we obtain the analytical solutions of the original heat transfer problem by superimposing the Eqs. (10) and (25)

$$T(z,t)=A_r z+B_r+\sum_{n=1}^{\infty}\frac{\varphi_n(z)}{N(\beta_n)}e^{-\kappa\beta_n^2 t}\left[\bar{F}^*(\beta_n)+\int_0^t e^{\kappa\beta_n^2\tau}A(\beta_n,\tau)\,d\tau\right] \qquad -l<z<l,\ t>0 \tag{26}$$

### B. Dynamic Thermal Stress Problem

Under the above prescribed temperature distribution, thermal stresses and deformations will be induced in the elastic slab. Since the slab extends to infinity in the transverse direction and because of symmetry, there is no transverse motions, that is, the displacement in $x$ and $y$ direction are vanishing, and so the motion of the slab can be described only by the displacement in the $z$ direction, denoted by $w=w(z,t)$. The only nonzero strain component is $\varepsilon_{zz}=\partial w/\partial z$. Hence, the problem reduces to a one-dimensional strain plane wave propagation problem.

The stresses in terms of displacement are

$$\sigma_{zz}=(\lambda+2\mu)\frac{\partial w}{\partial z}-\beta\Gamma \qquad \sigma_{xx}=\sigma_{yy}=\lambda\frac{\partial w}{\partial z}-\beta\Gamma$$
$$\sigma_{xy}=\sigma_{yz}=\sigma_{zx}=0 \tag{27}$$

where

$$\Gamma=\Gamma(z,t)=T(z,t)-F(z) \qquad \lambda=\frac{\nu E}{(1+\nu)(1-2\nu)},$$
$$\mu=\frac{E}{2(1+\nu)}, \qquad \beta=\frac{\alpha E}{1-2\nu} \tag{28}$$

where $\Gamma(z,t)$ represents the changed temperature above initial condition. $\nu$ is Poisson's ratio. $E$ is Young's modulus. $\alpha$ is the coefficient of linear thermal expansion. $\lambda$ and $\mu$ are Lame's constant.

The one-dimensional thermoelastic equations of motion with the inertia term can be expressed as

$$\frac{\partial^2 w}{\partial z^2}-\frac{1}{\upsilon^2}\frac{\partial^2 w}{\partial t^2}=\chi\frac{\partial\Gamma}{\partial z} \qquad -l<z<l,\ t>0 \tag{29}$$

where

$$\upsilon^2=\frac{\lambda+2\mu}{\rho}, \qquad \chi=\frac{1+\nu}{1-\nu}\alpha \tag{30}$$

$\upsilon$ being the velocity of irrotational wave.

If the top and bottom surfaces are both traction-free, then the boundary conditions can be expressed as

$$(\lambda+2\mu)\frac{\partial w}{\partial z}-\beta\Gamma(z,t)=0, \quad \text{at } z=-l \tag{31a}$$

$$(\lambda+2\mu)\frac{\partial w}{\partial z}-\beta\Gamma(z,t)=0, \quad \text{at } z=l \tag{31b}$$

The initial conditions are assumed as

$$w(z,t)=0, \qquad \frac{\partial w}{\partial t}=0, \quad \text{at } t=0 \tag{32}$$

To homogenize the above problem, we divide the solution of $w(z,t)$ into two parts as following

$$w(z,t)=W(z,t)+V(z,t) \qquad -l<z<l,\ t>0 \tag{33}$$

where $V(z,t)$ satisfy the following inhomogeneous boundary conditions

$$(\lambda+2\mu)\frac{\partial V}{\partial z}-\beta\Gamma(z,t)=0, \quad \text{at } z=-l \tag{34a}$$

$$(\lambda + 2\mu)\frac{\partial V}{\partial z} - \beta\Gamma(z,t) = 0, \quad \text{at } z = l \tag{34b}$$

In view of the above conditions, $V(z,t)$ can be assumed in the form

$$V(z,t) = A_v z^2 + B_v z \tag{35}$$

where rigid body displacement have been eliminated in this solution, and

$$A_v = \frac{\beta[\Gamma(l,t) - \Gamma(-l,t)]}{4l(\lambda + 2\mu)}, \qquad B_v = \frac{\beta[\Gamma(l,t) + \Gamma(-l,t)]}{2(\lambda + 2\mu)} \tag{36}$$

On the other hand, $W(z,t)$ satisfies the following inhomogeneous thermoelastic equation of motion

$$\frac{\partial^2 W}{\partial z^2} + G(z,t) = \frac{1}{\upsilon^2}\frac{\partial^2 W}{\partial t^2} \qquad -l < z < l, t > 0 \tag{37}$$

where

$$G(z,t) = \frac{\partial^2 V}{\partial z^2} - \frac{1}{\upsilon^2}\frac{\partial^2 V}{\partial t^2} - \chi\frac{\partial \Gamma}{\partial z} \tag{38}$$

with the following homogeneous boundary conditions

$$\frac{\partial W}{\partial z} = 0, \quad \text{at } z = -l \tag{39a}$$

$$\frac{\partial W}{\partial z} = 0, \quad \text{at } z = l \tag{39b}$$

and the initial condition

$$W(z,t) = -V(z,t), \qquad \frac{\partial W}{\partial t} = -\frac{\partial V}{\partial t}, \quad \text{at } t = 0 \tag{40}$$

Now define the integral transform pair in the space variable $z$ for the solution $W(z,t)$ as follows:

Integral transform

$$\bar{W}(\gamma_m, t) = \int_{-l}^{l} \phi_m(z) W(z,t)\, \mathrm{d}z \tag{41a}$$

Inversion formula

$$W(z,t) = \sum_{m=1}^{\infty} \frac{\phi_m(z)}{N(\gamma_m)} \bar{W}(\gamma_m, t) \tag{41b}$$

where eigenfunctions $\phi_m(z)$ satisfy the following eigenvalue problem:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} + \gamma^2\phi(z) = 0 \qquad -l < z < l \tag{42}$$

with the following boundary conditions:

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = 0, \quad \text{at } z = -l \tag{43a}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = 0, \quad \text{at } z = l \tag{43b}$$

The solutions of the above eigenvalue problem are

$$\phi_m(z) = \cos\gamma_m(z + l) \tag{44}$$

where

$$\gamma_m = \frac{m\pi}{2l}, \qquad m = 1, 2 \ldots \tag{45}$$

The eigenfunction (44) satisfies the following orthogonality relation:

$$\int_{-l}^{l} \phi_m(z)\phi_n(z)\, \mathrm{d}z = \begin{cases} 0 & m \neq n \\ N(\gamma_m) & m = n \end{cases} \tag{46}$$

where the norm $N(\gamma_m)$ is defined as

$$N(\gamma_m) = \int_{-l}^{l} [\phi_m(z)]^2\, \mathrm{d}z \tag{47}$$

We now apply the transformation (41a) to the differential Eq. (37). From the boundary and initial conditions (39) and (40) and by the mathematical derivation, this transformation finally leads to

$$\frac{\mathrm{d}^2\bar{W}}{\mathrm{d}t^2} + \upsilon^2\gamma_m^2\bar{W}(\gamma_m, t) = A(\gamma_m, t) \qquad t > 0 \tag{48}$$

where

$$A(\gamma_m, t) = \upsilon^2 \int_{-l}^{l} \phi_m(z) G(z,t)\, \mathrm{d}z \tag{49}$$

with the following initial conditions

$$\bar{W}(\gamma_m, t) = \bar{V}_0(\gamma_m), \quad \text{at } t = 0 \qquad \frac{\mathrm{d}\bar{W}}{\mathrm{d}t} = \bar{V}_1(\gamma_m), \quad \text{at } t = 0 \tag{50}$$

where

$$\begin{aligned} \bar{V}_0(\gamma_m) &= -\int_{-l}^{l} \phi_m(z) V(z,t)\, \mathrm{d}z, \quad \text{at } t = 0 \\ \bar{V}_1(\gamma_m) &= -\int_{-l}^{l} \phi_m(z)\frac{\partial V}{\partial t}\, \mathrm{d}z, \quad \text{at } t = 0 \end{aligned} \tag{51}$$

The solution of Eq. (48) subjected to the initial condition (50) gives

$$\bar{W}(\gamma_m, t) = K_m \cos(\upsilon\gamma_m t) + L_m \sin(\upsilon\gamma_m t) \tag{52}$$

where

$$\begin{aligned} K_m &= \bar{V}_0(\gamma_m) - \frac{1}{\upsilon\gamma_m}\int_0^t \sin(\upsilon\gamma_m\tau) A(\gamma_m, \tau)\, \mathrm{d}\tau \\ L_m &= \frac{1}{\upsilon\gamma_m}\left[\bar{V}_1(\gamma_m) + \int_0^t \cos(\upsilon\gamma_m\tau) A(\gamma_m, \tau)\, \mathrm{d}\tau\right] \end{aligned} \tag{53}$$

By introducing this integral transform into the inversion formula (41b), we obtain

$$W(z,t) = \sum_{m=1}^{\infty} \frac{\phi_m(z)}{N(\gamma_m)}[K_m \cos(\upsilon\gamma_m t) + L_m \sin(\upsilon\gamma_m t)] \tag{54}$$

Finally, we obtain the analytical solution of displacement of the dynamic thermal stress problem by superimposing the Eqs. (35) and (54):

$$\begin{aligned} w(z,t) = {} & A_v z^2 + B_v z + \sum_{m=1}^{\infty} \frac{\phi_m(z)}{N(\gamma_m)}[K_m \cos(\upsilon\gamma_m t) \\ & + L_m \sin(\upsilon\gamma_m t)] \qquad -l < z < l, t > 0 \end{aligned} \tag{55}$$

Substituting the above result into Eq. (27), the analytical solutions for the dynamic thermal stresses are expressed as

$$\begin{aligned} \sigma_{zz} = {} & (\lambda + 2\mu)\left\{2A_v z + B_v + \sum_{m=1}^{\infty} \frac{1}{N(\gamma_m)}\frac{\mathrm{d}\phi_m(z)}{\mathrm{d}z}[K_m \cos(\upsilon\gamma_m t) \right. \\ & \left. + L_m \sin(\upsilon\gamma_m t)]\right\} \\ & - \beta\Gamma(z,t) \qquad -l < z < l, t > 0 \end{aligned} \tag{56a}$$

and

$$\sigma_{xx} = \sigma_{yy} = \lambda\left\{2A_v z + B_v + \sum_{m=1}^{\infty}\frac{1}{N(\gamma_m)}\frac{d\phi_m(z)}{dz}[K_m\cos(\upsilon\gamma_m t) + L_m\sin(\upsilon\gamma_m t)]\right\} - \beta\Gamma(z,t) \qquad -l < z < l, t > 0 \tag{56b}$$

### C. Validation of the Proposed Method

To validate the proposed method, a general thermal shock problem is considered. For this example, the nickel-based superalloy is used and the thickness of slab is selected as 0.8 $\mu$m for the sake of reducing computation cost in the ABAQUS (a commercial finite element software) simulation; the slab is suddenly exposed to a convective medium of temperature 1000°C with a heat transfer coefficient of $2.5453 \times 10^7$ W m$^{-2}$ K$^{-1}$. Then the results predicted by the proposed method are compared with those achieved by Lu and Fleck [8], and an ABAQUS simulation, as shown in Figs. 2 and 3. In the legend, Present indicates the prediction by our model. The transient temperature of top surface is illustrated in Fig. 2, which shows our result is consistent with both ABAQUS simulation and Lu and Fleck's result [8]. The dynamical thermal stress distribution after 30 ps is shown in Fig. 3. It is found that our results of out-plane stress $\sigma_{zz}$ and in-plane stress $\sigma_{xx}$ well agree with the ABAQUS simulation.

## III. Application in Hypersonic Flight Environment

Now the above theory is applied to aerodynamic heating case. For determining the temperatures, aerodynamic heating is often represented as a convective boundary condition. Boundary-layer analysis of compressible flow shows that the heat transfer $q_h$ (positive into the surface) between the boundary layer and the top surface of slab can be expressed as [12]

$$q_h = \zeta_1(T_r - T_s) \tag{57}$$

where $T_r$ represents the recovery temperature or the adiabatic wall temperature, and $T_s$ is the top surface temperature. Extensive studies of the flat-plate problem [13] show that $T_r$ can be calculated with good accuracy by

$$T_r = T_\infty\left(1 + r\frac{\gamma - 1}{2}M_\infty^2\right) \tag{58}$$

where $r$ is the recovery factor, $T_\infty$ and $M_\infty$ are the temperature and Mach number of the freestream, respectively. Also, $\gamma$ denotes the ratio of specific heats $\gamma = c_p/c_v$, where $c_v$ is specific heat at constant volume. $M_\infty$ may be varied with time, to represent a Mach number curve for hypersonic flight. As a preliminary study, heat transfer by radiation to or from the top surface is not considered.

To protect the aerospace vehicle structure from the severe aerodynamic heating and keep structural temperatures within acceptable ranges, many leading edges and panels generally require active cooling systems, comprising coolant-cooled through internal channels or from the inside surfaces [12,14]. The latter is considered in this paper, and is also represented as a convective boundary condition. Hence, the heat flux $q_c$ (positive into the surface) transferred between the coolant and the bottom surface of slab may be expressed as

$$q_c = \zeta_2(T_s - T_c) \tag{59}$$

where $T_c$ is the temperature of the coolant.

In view of boundary conditions (3), (57), and (59), we know

$$f_1(t) = \zeta_1 T_r \tag{60a}$$

$$f_2(t) = \zeta_2 T_c \tag{60b}$$

Then by substituting these equations into Eqs. (26), (55), and (56), we can obtain the analytical solutions to simulate the dynamic thermoelastic response of our present system during hypersonic flight.

## IV. Results and Discussion

As an illustrative example, computations are carried out on an infinite UHTC-based slab with 3 mm thickness in hypersonic flight environment at various Mach numbers. The initial temperature is assumed to be 295 K and the temperature of the fluid at infinite distance is 243 K. The coolant is assumed to keep a specified temperature of 288 K. The values of convective heat transfer on the aerodynamic surface and coolant surface are selected as $2.94 \times 10^3$ W m$^{-2}$ K$^{-1}$ [15] and $1.82 \times 10^4$ W m$^{-2}$ K$^{-1}$ [12], respectively.

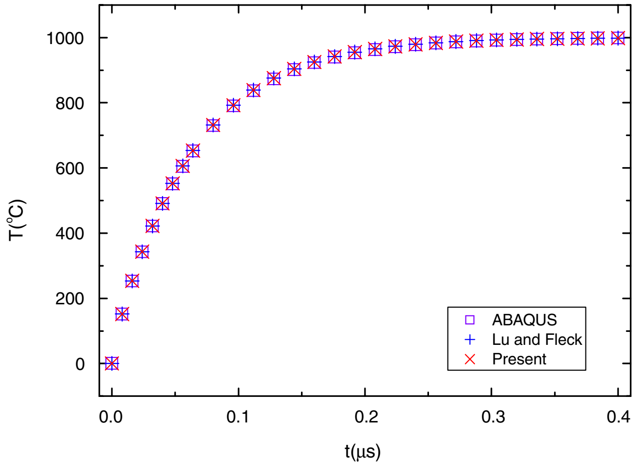


**Fig. 2 Transient temperature as a function of time on the top surface of the slab.**

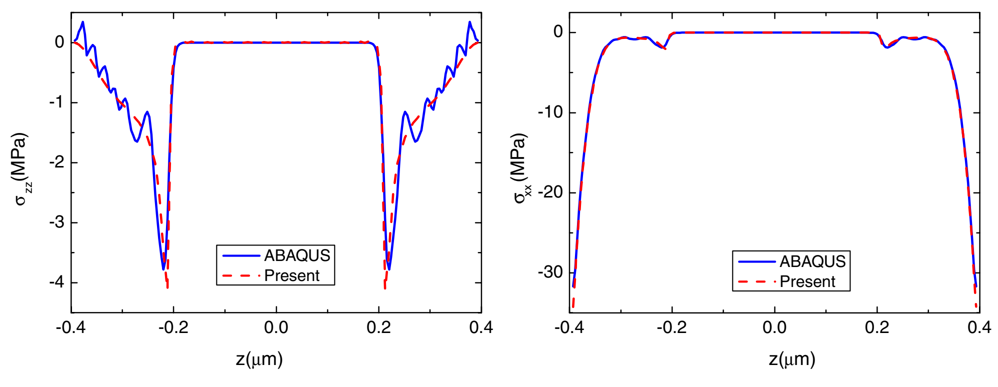


**Fig. 3 Dynamic thermal stress distribution along the thickness direction of the slab at 30 ps for a) out-plane stress $\sigma_{zz}$, and b) in-plane stress $\sigma_{xx}$.**

**Table 1 Properties of $ZrB_2$ used in the computation**

| Material | $k$ (W m$^{-1}$ K$^{-1}$) | $c_p$ (J kg$^{-1}$ K$^{-1}$) | $\alpha$ ($10^{-6}$ K$^{-1}$) | $\rho$, kg m$^{-3}$ | $E$, GPa | $\nu$ |
|---|---|---|---|---|---|---|
| $ZrB_2$ | 60 | 427 | 5.9 | 6119 | 489 | 0.15 |

The UHTC material is $ZrB_2$, whose properties are tabulated in Table 1 [16].

Figure 4 shows the distribution and propagation of stress wave on the normal section along thickness for several times in the characteristic time $t_0 = l/\upsilon$. In this case, Mach number is taken as 5. It is found that the normal stress $\sigma_{zz}$ spreads inward both from the upper and lower surface as time elapses. Moreover, since the different thermal boundary conditions on these surfaces, very strong shock waves are induced on aerodynamic surface, while moderate shock waves are generated on the coolant surface. As well, the peak of normal stress $\sigma_{zz}$ is at the order of 1500 Pa, which is much smaller than the in-plane stress shown in the next figures.

The variations in temperature and in-plane stresses on the aerodynamic surface and coolant surface versus time under various Mach numbers are shown in Figs. 5 and 6, respectively. In this case, Mach number is ranged from 3 to 10. It is concluded that the temperature and stress increase rapidly as Mach number increases.

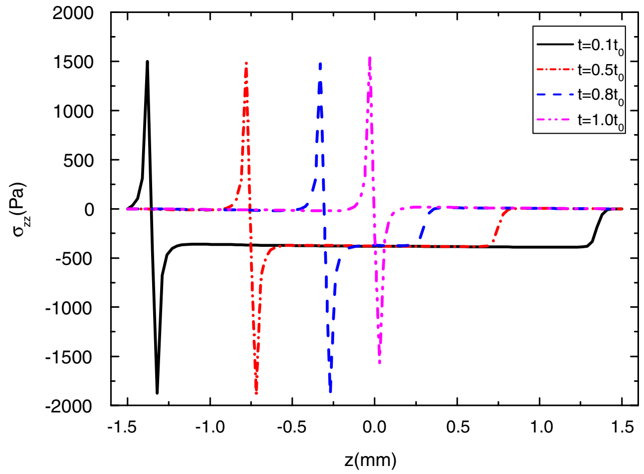


**Fig. 4 Distribution and propagation of stress wave.**

## V. Conclusions

In this paper, dynamic thermoelastic response in an infinite elastic slab with finite thickness is studied. The system of generalized governing equations with generalized initial and boundary conditions is solved by the method of finite integral transformation and the analytic expressions of the transient temperature and stresses in the slab are obtained. Moreover, the analytical solutions are extended to simulate the dynamic thermoelastic response of our present system during hypersonic flight. Calculations are carried out for UHTC (zirconium diboride) samples to provide detailed information of temperature and stresses induced by aerodynamic heating. The results lead to the following conclusions. First, it is found that the severe aerodynamic heating induces strong thermal shock and in-plane stress in the slab. Second, the effects of Mach number on the variations in temperature and stresses are distinct, and the temperature and stress increase rapidly as Mach number increases.

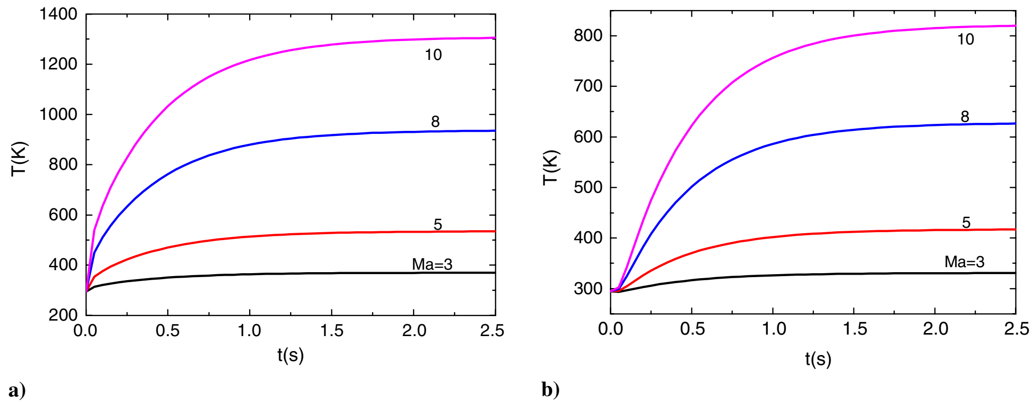


**Fig. 5 Temperature variation with time on the a) aerodynamic surface, and b) coolant surface.**

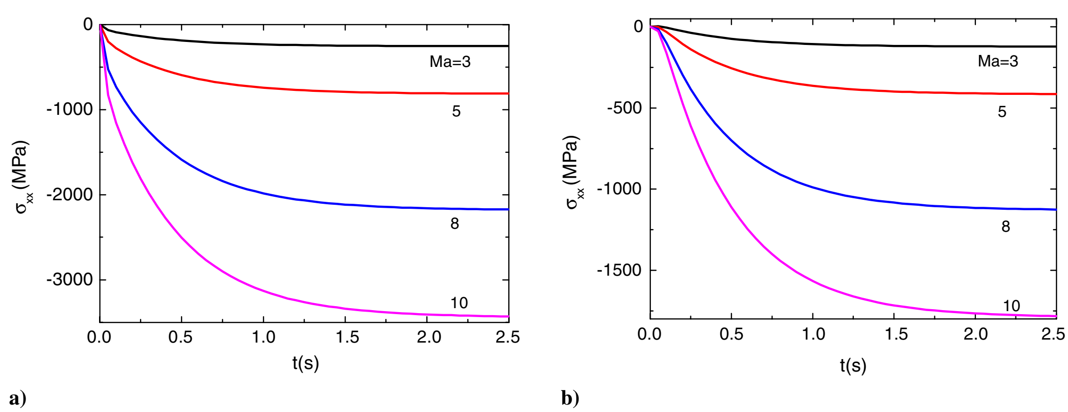


**Fig. 6 In-plane stress variation with time on the a) aerodynamic surface, and b) coolant surface.**

The model as it stands now, studies a slab with finite thickness, although it has been considerably simplified from the real case to obtain analytical solutions. Actually, this method can be extended to consider for column and sphere-cone geometry. In particular, the future work will help us to understand in more detail the influence of complicated geometry, the effects of temperature dependence of the thermal and physical properties and the thermal radiation effects, which involves nonlinear and is expected to be done by numerical simulation technique.

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